

Contents lists available at [SciVerse ScienceDirect](http://SciVerse.ScienceDirect.com)

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

An analysis of inverse source problems with final time measured output data for the heat conduction equation: A semigroup approach

Alemdar Hasanov^a, Marián Slodička^{b,*}^a Department of Mathematics and Computer Sciences, Izmir University, 35350 Uckuyular, Izmir, Turkey^b Department of Mathematical Analysis, Ghent University, Galglaan, 2, B-9000 Ghent, Belgium

ARTICLE INFO

Article history:

Received 4 June 2012

Received in revised form 20 August 2012

Accepted 21 August 2012

Keywords:

Inverse source problem

Semigroup approach

Final overdetermination

Representation of a solution

Uniqueness

Fourier method

ABSTRACT

This paper presents a semigroup approach for inverse source problems for the abstract heat equation $u_t = Au + F$, when the measured output data is given in the form the final overdetermination $u_T(x) := u(x, T)$. A representation formula for a solution of the inverse source problem is proposed. This representation shows a non-uniqueness structure of the inverse problem solution, and also permits one to derive a sufficient condition for uniqueness. Some examples related to identifying the unknown spacewise and time-dependent heat sources $f(x)$ and $h(t)$ of the heat equation $u_t = u_{xx} + f(x)h(t)$, from the final overdetermination or from a single point time measurement are presented.

© 2012 Elsevier Ltd. All rights reserved.

1. Introduction

It is well-known that semigroup notion is one of most important tools for describing time-dependent processes in nature in terms of functional analysis (see, [1–9]). The key relations here are $S(t + \tau) = S(t)S(\tau)$, where $t, \tau \in [0, \infty)$ are time parameters, and $S(0) = I$. The semigroup $\{S(t)\}_{t \geq 0}$ on a Banach space \mathcal{B} is defined to be as a family of operators $S(t) : \mathcal{B} \mapsto \mathcal{B}$, for all $t \in [0, \infty)$, such that the above two conditions hold. If $A : \mathcal{D}(A) \subset \mathcal{B} \mapsto \mathcal{B}$ is a closed and densely defined linear operator, it can be shown that for a given element $u_0 \in \mathcal{B}$, the function $u : [0, \infty) \mapsto \mathcal{B}$, defined to be as $u(t) := S(t)u_0$, is the unique solution of the Cauchy problem $u'(t) = Au(t)$, $t > 0$; $u(0) = u_0$, for the abstract heat equation $u'(t) = Au(t)$. The operator $A : \mathcal{D}(A) \mapsto \mathcal{B}$, with the domain $\mathcal{D}(A) := \{u \in \mathcal{B} : \lim_{t \rightarrow 0^+} (S(t)u - u)/t\}$ and defined to be as $Au := \lim_{t \rightarrow 0^+} (S(t)u - u)/t$, $u \in \mathcal{B}$, is called the infinitesimal generator of the semigroup $\{S(t)\}_{t \geq 0}$.

The first attempt to study source identification problems for the time independent source $F_0 \in \mathcal{B}$ parabolic equation $u'(t) = Au(t) + F_0$, with the final overdetermination $u_T(x) := u(x, T)$, by the semigroup approach has been given in [10]. It is proved here that when the elliptic operator $-A$ is positive definite and self-adjoint, the solution $\langle u, F_0 \rangle$ of the source identification problem exists and is unique. A general representation formula for a solution of the source identification problem for the abstract parabolic equation $u'(t) = Au(t) + F(t)$ with time-dependent source $F(t)$, has been proposed in [11]. Note that an inverse source problem with final overdetermination for the one dimensional heat equation has first been considered by Tikhonov [12] in study of geophysical problems. In this work the heat equation with prescribed lateral and final data is studied in half-plane and the uniqueness of the bounded solution is proved. For parabolic equations in a bounded domain, various aspects of inverse source problems has been studied in [13–16], etc.

* Corresponding author.

E-mail addresses: alemdar.hasanoglu@izmir.edu.tr (A. Hasanov), marian.slodicka@ugent.be (M. Slodička).URLs: <http://www.izmir.edu.tr/ahasanoglu/> (A. Hasanov), <http://cage.ugent.be/~ms/> (M. Slodička).

In this study a general representation formula for the solution of inverse source problems (ISPs) for the abstract parabolic equation, as well as for the heat equation, are proposed. This representation formula permits one to derive a structure of a solution, and also uniqueness cases. Some applications to the heat equation with separated spacewise and time-dependent sources are illustrated.

2. Representation formula for a solution of the ISP with final overdetermination

Let $A : \mathcal{D}(A) \subset \mathcal{B} \mapsto \mathcal{B}$ be the infinitesimal generator of the strongly continuous semigroup $\{S(t)\}_{t \geq 0}$. Then the domain $\mathcal{D}(A)$ of A is dense in \mathcal{B} , and A is a closed operator, i.e. for all $\{u_n\} \subset \mathcal{D}(A)$, $u_n \rightarrow u$, $Au_n \rightarrow z$ implies $z = Au$ (see, [8, Theorem 11.12]). Consider the abstract Cauchy problem

$$\begin{cases} u'(t) = Au(t) + F(t), & t \in (0, T], \\ u(0) = u_0, \end{cases} \quad (1)$$

assuming that $u_0 \in \mathcal{D}(A)$ and $F \in C([0, T]; \mathcal{B})$. Then the solution $u \in C^1([0, T]; \mathcal{B}) \cap C([0, T]; \mathcal{D}(A))$ of the Cauchy problem (1) can be represented by the formula

$$u(t) = \mathcal{S}(t)u_0 + \int_0^t \mathcal{S}(t - \tau)F(\tau)d\tau, \quad t \in (0, T]. \quad (2)$$

Due to the continuity of the source $F(t)$, this solution is defined to be the classical solution, since the mapping $t \mapsto S(t)Au$ is differentiable for each $t > 0$, and

$$\frac{d}{dt} \mathcal{S}(t)u = A\mathcal{S}(t)u = \mathcal{S}(t)Au, \quad t \in (0, T], \quad u \in \mathcal{D}(A).$$

The representation formula makes sense also under weaker conditions, if $u_0 \in \mathcal{B}$ and $F \in L_1([0, T]; \mathcal{B})$. Note that this solution, defined to be the mild solution of (1), corresponds to the weak solution of the corresponding parabolic problem, when $\mathcal{B} = H^0(\Omega)$, and the elliptic operator $-A$ with the domain $\mathcal{D}(A) := \dot{H}^1(\Omega) \cap H^2(\Omega)$, is given by

$$Au = -\frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial u}{\partial x_j} \right) + b_i(x) \frac{\partial u}{\partial x_i} + c(x)u, \quad x \in \Omega \subset \mathbb{R}^n. \quad (3)$$

Let us now formulate the *abstract inverse source problem* (subsequently, *AISP*). Assume that $F \in \mathcal{F} \subset L_1([0, T]; \mathcal{B})$ and \mathcal{F} be the set of admissible sources. AISP consists of determining the unknown source term $F \in \mathcal{F}$ in the Cauchy problem (1) from the with final overdetermination (measured output data) $u_T \in \mathcal{B}$ defined to be as

$$u_T := u(t)|_{t=T}, \quad T > 0. \quad (4)$$

In this context, the Cauchy problem (1) will be regarded as a *direct problem*.

We denote by $u(t; F)$ the unique solution of the direct problem (1), corresponding to the given source term $F \in \mathcal{F}$. Then introducing the input–output map $\Phi : \mathcal{F} \mapsto \mathcal{B}$, $\Phi F := u(t; F)|_{t=T}$, $T > 0$, can reformulate AISP as the following operator equation: $\Phi F = u_T$, $F \in \mathcal{F}$, $u_T \in \mathcal{B}$. Thus the considered inverse problem can be reduced to the problem of invertibility of the input–output map Φ .

Let us substitute $t = T$ in the semigroup representation (2) and use the additional condition (4). Then we have:

$$\int_0^T \mathcal{S}(T - \tau)F(\tau)d\tau = u_T - \mathcal{S}(T)u_0, \quad T > 0. \quad (5)$$

A solution of the integral equation (5) will be defined as a *mild solution* (or simply, solution) of AISP. If $\mathcal{F} \subset W^{1,1}([0, T]; \mathcal{B}')$, then this solution will be also the classical solution of the abstract Cauchy problem (1) (see, [8, Theorem 11.16]).

The principal tool in the analysis of the solution of AISP is the following representation formula.

Theorem 1. Let $u_0, u_T \in \mathcal{D}(A)$. Assume that $S(t)$ is a uniformly continuous semigroup of the elliptic operator $-A$. Then for any function $g \in H^0(0, T; \mathcal{D}(A))$, the function

$$\tilde{F}(t) = A(\mathcal{S}(T) - I)^{-1}(u_T - \mathcal{S}(T)u_0) - A(\mathcal{S}(T) - I)^{-1} \int_0^T \mathcal{S}(T - \tau)g(\tau)d\tau + g(t), \quad (6)$$

$t \in (0, T]$, is a solution of AISP, defined by (1) and (4).

Conversely, if $\tilde{F}(t) \in \mathcal{F}$ is any solution of AISP, then there exists such a function $g \in H^0(0, T; \mathcal{D}(A))$ that this solution can be represented by formula (6).

Proof. The operator $-A$ is elliptic, thus A is dissipative. Using [7, Chapter 7.3] and the Lumer–Phillips Theorem (see, [7, Chapter 1.4]) we see that A generates a contractive semigroup. Then following the Lumer–Phillips Theorem we conclude the existence of $(\mathcal{S}(T) - I)^{-1}$.

Let us substitute the function $\tilde{F}(t)$, given by (5), on the left hand side integral of (4), and calculate it. Then we have:

$$\begin{aligned} \int_0^T \mathcal{S}(T-\tau) \tilde{F}(\tau) d\tau &= \int_0^T \mathcal{S}(T-\tau) A(\mathcal{S}(T) - I)^{-1} (u_T - \mathcal{S}(T)u_0) d\tau - \int_0^T \mathcal{S}(T-\tau) A(\mathcal{S}(T) - I)^{-1} d\tau \\ &\quad \times \int_0^T \mathcal{S}(T-\tau) g(\tau) d\tau + \int_0^T \mathcal{S}(T-\tau) g(\tau) d\tau. \end{aligned}$$

We use the identity (see, [8, Lemma 11.11])

$$\int_0^t \mathcal{S}(\tau) v d\tau = A^{-1} \int_0^t \partial_\tau e^{A\tau} v d\tau = A^{-1} (\mathcal{S}(t) - I) v, \quad \forall v \in \mathcal{D}(A), \quad t \in (0, T] \quad (7)$$

in the first and second right hand side integrals assuming $v \in \mathcal{D}(A)$ and substituting in (7) $t = T$. Then we obtain:

$$\int_0^T \mathcal{S}(T-\tau) \tilde{F}(\tau) d\tau = u_T - \mathcal{S}(T)u_0 - I \cdot \int_0^T \mathcal{S}(T-\tau) g(\tau) d\tau + \int_0^T \mathcal{S}(T-\tau) g(\tau) d\tau.$$

This shows that the function $\tilde{F}(t)$, given by (5), is the solution of the integral equation (5).

To prove the second part of the theorem, now assume that $F(t)$ is a solution of the integral equation (5). We introduce the function

$$g(t) = F(t) - A(\mathcal{S}(T) - I)^{-1} (u_T - \mathcal{S}(T)u_0) + v, \quad (8)$$

where the arbitrary function $v \in \mathcal{D}(A)$ will be defined below. Acting by the semigroup operator $\mathcal{S}(T - \tau)$ to the both sides of (8) and then integrating on $[0, T]$ we get:

$$\begin{aligned} \int_0^T \mathcal{S}(T-\tau) g(\tau) d\tau &= \int_0^T \mathcal{S}(T-\tau) F(\tau) d\tau - \int_0^T \mathcal{S}(T-\tau) A(\mathcal{S}(T) - I)^{-1} (u_T - \mathcal{S}(T)u_0) d\tau \\ &\quad + \int_0^T \mathcal{S}(T-\tau) v d\tau. \end{aligned}$$

The first right hand side integral here is $u_T - \mathcal{S}(T)u_0$, according to (5). In the second and third right hand side integrals we use identity (7). Then we obtain:

$$\int_0^T \mathcal{S}(T-\tau) g(\tau) d\tau = u_T - \mathcal{S}(T)u_0 - A(\mathcal{S}(T) - I)^{-1} (u_T - \mathcal{S}(T)u_0) A^{-1} (\mathcal{S}(T) - I) + A^{-1} (\mathcal{S}(T) - I) v.$$

Hence the arbitrary element $v \in \mathcal{D}(A)$ is defined as follows:

$$v = A(\mathcal{S}(T) - I)^{-1} \int_0^T \mathcal{S}(T-\tau) g(\tau) d\tau.$$

Substituting this in (8) we obtain the required formula (6). This completes the proof. \square

3. An inverse source and backward problems with time independent source term

Consider the special case of AISP with the *time independent source term* $F(t) \equiv F_0, \forall t \in (0, T)$:

$$\begin{cases} u'(t) = Au + F_0, & t \in (0, T), \\ u(0) = u_0, & u(T) = u_T, \end{cases} \quad (9)$$

where $F_0 \in \mathcal{F}_0 \subset \mathcal{D}(A)$, and \mathcal{F}_0 is the set of time independent source terms. Substituting $F(t) = F_0$ in (5) we get:

$$\int_0^T \mathcal{S}(T-\tau) F_0 d\tau = u_T - \mathcal{S}(T)u_0, \quad T > 0. \quad (10)$$

By identity (7), this implies: $A^{-1} (\mathcal{S}(T) - I) F_0 = \int_0^T \mathcal{S}(\tau) F_0 d\tau = \int_0^T \mathcal{S}(T-\tau) F_0 d\tau = u_T - \mathcal{S}(T)u_0, \quad T > 0$. Hence in this case we have a unique solution (see, also [11]).

Lemma 1. *Let conditions of Theorem 1 hold. Then for any time independent source term $F_0 \in \mathcal{F}_0 \subset \mathcal{D}(A)$, the unique solution of the inverse problem (8) can be represented by the formula*

$$F_0 = A(\mathcal{S}(T) - I)^{-1} (u_T - \mathcal{S}(T)u_0). \quad (11)$$

Comparing this result with formula (6) we obtain the following result.

Corollary 1. Let conditions of [Theorem 1](#) hold. Then any solution of AISP with time-dependent source term \tilde{F} is the sum of two elements:

$$\tilde{F} = F_0 + F(t), \quad F_0 \in \mathcal{F}_0, \quad F \in \mathcal{F}, \quad (12)$$

where F_0 is the unique solution, given by (11), of ISP (9) with the time independent source term $F_0 \in \mathcal{D}(A)$, and $F \in \mathcal{F}$, given by

$$F(t) = g(t) - A(\mathcal{J}(T) - I)^{-1} \int_0^T \mathcal{J}(T - \tau)g(\tau)d\tau, \quad (13)$$

with an arbitrary function $g \in H^0(0, T; \mathcal{D}(A))$, is a solution of AISP

$$\begin{cases} u'(t) = Au + F(t), & t \in (0, T], \\ u(0) = 0, & u(T) = 0, \end{cases}$$

with the homogeneous initial and final data.

Formulas (11)–(13) show the structure of the representation formula (6) for a solution of AISP. Let us analyze now the uniqueness of the solution of the backward problem

$$\begin{cases} u'(t) = Au + F_0, & t \in (0, T], \\ u(T) = u_T, \end{cases} \quad (14)$$

for the abstract parabolic equation with a time independent source term.

At this point we would like to mention that this problem has already been studied in the literature, e.g. in [4–6], where Carleman estimates have been used. We bring a short proof for a general elliptic operator using the semigroup theory. Note that if A is elliptic, then it is also a sectorial operator and its generates an analytic semigroup.

Lemma 2. Let $u_T^{(1)}, u_T^{(2)} \in \mathcal{D}(A)$, be two final data and $u^{(1)}(t), u^{(2)}(t) \in H^0(0, T; V)$ be corresponding solutions to (14). If $u_T^{(1)} = u_T^{(2)}$, then $u^{(1)}(0) = u^{(2)}(0)$.

Proof. Denote by $v(t) := u^{(1)}(t) - u^{(2)}(t)$. Then $v(t)$ satisfies the homogeneous equation $v'(t) = Av$ and the final condition $v(T) = 0$. Applying formula (10) to this function we conclude $\mathcal{J}(T)v(0) = 0$. Now, we use the following moment inequality [17, Proposition 6.6.4],

$$\|B^\theta x\| \leq \frac{C}{\theta(1-\theta)} \|x\|^{1-\theta} \|Bx\|^\theta,$$

which is valid for any sectorial operator B . Recalling that any linear bounded operator is sectorial, we may set $B = \mathcal{J}(T)$, $x = v(0)$, $\theta = \frac{\epsilon}{T}$ to get

$$\|\mathcal{J}(t)v(0)\| \leq \frac{C}{\frac{\epsilon}{T}(1-\frac{\epsilon}{T})} \|\mathcal{J}(T)v(0)\|^{\frac{\epsilon}{T}} \|v(0)\|^{1-\frac{\epsilon}{T}}.$$

Therefore we have

$$\mathcal{J}(T)v(0) = 0 \implies \mathcal{J}(T/2)v(0) = 0.$$

Using this recursion and the continuity of $\mathcal{J}(t)$ we see that

$$0 = \lim_{k \rightarrow \infty} \mathcal{J}\left(\frac{T}{2^k}\right)v(0) = \mathcal{J}(0)v(0) = v(0),$$

which concludes the proof. \square

4. Inverse source problems in the case of separated variables source terms

Consider now AISP

$$\begin{cases} u'(t) = Au + F_0H(t), & t \in (0, T), \\ u(0) = u_0, & u(T) = u_T, \end{cases} \quad (15)$$

with the separated variables source terms $F(t) := F_0H(t)$, where $F_0 \in \mathcal{F}_0 \subset \mathcal{D}(A)$ is the time independent and $H \in \mathcal{H} \subset C[0, T]$ is the time-dependent source terms.

We define here two ISPs: the problem of identification the unknown time independent source $F_0 \in \mathcal{F}_0$, when the time-dependent source term $H \in \mathcal{H}$ is known (subsequently, the problem AISP(F)), and the problem of identification the unknown time-dependent source term $H \in \mathcal{H}$, when $F_0 \in \mathcal{F}_0$ is unknown (subsequently, the problem AISP(H)).

Consider first the problem AISP(F). Substituting $F(t) = F_0 H(t)$ in (5) we get:

$$\int_0^T \mathcal{J}(T - \tau) F_0 H(\tau) d\tau = u_T - \mathcal{J}(T) u_0, \quad T > 0.$$

This implies:

$$\left(\int_0^T \mathcal{J}(T - \tau) H(\tau) d\tau \right) F_0 = u_T - \mathcal{J}(T) u_0, \quad T > 0.$$

Hence the unique solution of AISP(F), defined by (14), is obtained as follows:

$$F_0 = \left(\int_0^T \mathcal{J}(T - \tau) H(\tau) d\tau \right)^{-1} (u_T - \mathcal{J}(T) u_0). \quad (16)$$

To analyze the above problems AISP(H), we start with the simple example.

Example 1. Let $\Omega := (0, 1)$, and $A : D(A) \rightarrow \mathcal{B}$, $\mathcal{B} = H^0(\Omega) \equiv L_2(\Omega)$ is defined to be as,

$$A = -\frac{d^2}{dx^2}, \quad \mathcal{D}(A) := \dot{H}^1(\Omega) \cap H^2(\Omega),$$

where $\dot{H}^1(\Omega) := \{u \in H^1(\Omega) : u(0) = u(1) = 0\}$. The spectrum $\sigma(A)$ of A consists of the eigenvalues $\lambda_n = \pi^2 n^2$, $n \in \mathbb{N}$. The set $\{e_n(x)\}$ of corresponding eigenfunctions $e_n(x) = \sqrt{2} \sin(n\pi x)$, $n \in \mathbb{N}$, $x \in (0, 1)$, is an orthonormal complete system in $H^0(\Omega)$, i.e., $(e_n, e_m)_{H^0(\Omega)} = \delta_{nm}$, where δ_{nm} is the Kronecker delta. Using the method of separation of variables we see that

$$u(t) = e^{-At} u_0 := \sum_{n=1}^{\infty} e^{-\lambda_n t} (e_n, u_0) e_n$$

is the unique solution to the Cauchy problem $u_t + Au = 0$, $u(0) = u_0$, along with the homogeneous Dirichlet boundary conditions.

Now, for a given $f(x) \in H^0(0, 1)$ we need to find the pair $\langle u(x, t), h(t) \rangle$ in the problem

$$\begin{cases} u_t - u_{xx} = f(x)h(t), & (x, t) \in \Omega_T; \\ u(0, t) = u(1, t) = 0, & t \in (0, T); \\ u(x, 0) = 0, & x \in (0, 1), \end{cases} \quad (17)$$

by imposing an additional condition below. The question here is, *which additional data on $u(x, t)$ will ensure the uniqueness of the time-dependent source $h(t)$?*

The solution $u(x, t)$ of the parabolic (direct) problem (17) can be interpreted in the following form

$$u(x, t) = \int_0^t e^{-A(t-s)} h(s) f ds = \sum_{m=1}^{\infty} f_m \left\{ \int_0^t e^{-\lambda_m(t-s)} h(s) ds \right\} e_m(x), \quad f_m = (f, e_m)_{H^0(0,1)}. \quad (18)$$

For determination of the unknown source $h(t) \in H^0(0, T]$, let us first assume that *the measured output data is given in the form of the final overdetermination*: $u_T(x) := u(x, T)$, $T > 0$. Substituting $t = T$ in (18), multiplying both sides of (18) by $e_n(x)$, and then integrating on $[0, 1]$ we conclude:

$$\int_0^T e^{-\lambda_n(T-t)} h(t) dt = u_{T,n} / f_n, \quad \forall n \in \mathbb{N}, \quad u_{T,n} = (u_T, e_n)_{H^0(0,1)}, \quad n \in \mathbb{N}, \quad (19)$$

where $u_{T,n}$ are the Fourier coefficients of the function $u_T(x)$. Evidently, if the function $h(t) \in H^0(0, T]$ is a solution of the linear Fredholm equation of the first kind (19), then for any function $h_0(t) \in H^0(0, T]$, satisfying the conditions $(\exp(\lambda_n t), h_0)_{H^0[0,T]} = 0$, $n = 1, 2, 3, \dots$, the function $\tilde{h}(t) := h(t) + h_0(t) \in H^0(0, T]$ is also a solution of Eq. (19). Thus, the measured output data $u_T(x) := u(x, T)$, cannot uniquely determine the time-dependent unknown source $h(t)$, which means that the final overdetermination is not an optimal choice to ensure the uniqueness of $h(t)$.

To give a small hint: Consider $f(x) = e_1(x)$. Using the consideration above we see that

$$\int_0^T e^{\lambda_1 t} h(t) dt = 0.$$

Using the decomposition $L_2(0, T) = [e^{\lambda_1 t}] \oplus [e^{\lambda_1 t}]^\perp$ we see that this identity is fulfilled for any $h \in [e^{\lambda_1 t}]^\perp$. \square

We can generalize now this special result to the case when $\Omega \subset \mathbb{R}^n$, $n \geq 1$ is a bounded connected domain with the boundary $\partial\Omega$. Consider the linear differential operator A of second order, defined by (3). The aim is to study the problem of identification the pair $\langle u, H \rangle$ in the problem

$$\begin{cases} u_t(x, t) + Au(x, t) = F_0(x)H(t), & (x, t) \in \Omega_T := \Omega \times (0, T], \\ u(x, t) = 0, & (x, t) \in \partial\Omega \times (0, T), \\ u(x, 0) = u_0(x), & x \in \Omega; \\ u(x_0, t) = g(t), & t \in (0, T), x_0 \in \Omega, \\ u(x, T) = u_T(x), & x \in \Omega, \end{cases} \quad (20)$$

with the *separated variables source term* $F(x, t) := F_0(x)H(t)$, where $F_0 \in \mathcal{F}_0 \subset \mathcal{D}(A)$ and $H \in H^0(0, T)$. The $g(t)$ and $u_T(x)$ here are defined to be the measured output data. We adopt the ellipticity of the operator A in the space $\dot{H}^1(\Omega)$, assuming that $(Av, v) \geq \gamma_0^2 \|v\|_{\dot{H}^1(\Omega)}^2$, $\forall v \in \dot{H}^1(\Omega)$, $c(x) \geq 0$, $x \in \Omega$. Hence A is a sectorial operator, and $-A$ generates an analytical semigroup e^{-At} , $t \geq 0$.

Theorem 2. Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$ be a bounded domain with $\partial\Omega \in C^2$, $a_{ij} = a_{ji} \in C^1(\overline{\Omega})$, $b_i \in C(\overline{\Omega})$, $0 \leq c \in C(\overline{\Omega})$, $0 \leq F_0 \in C(\overline{\Omega})$, $F_0 \neq 0$ for all indices i and j . Let $\langle u^{(k)}, H^{(k)} \rangle$, $k = 1, 2$, be two solutions of problem (20). Assume that A is a uniformly elliptic operator. If $x_0 \in \Omega$, $g^{(1)}(t) = g^{(2)}(t)$ and $u_T^{(1)}(x) = u_T^{(2)}(x)$, then $\langle u^{(1)}, H^{(1)} \rangle = \langle u^{(2)}, H^{(2)} \rangle$, for all $(x, t) \in \Omega_T$.

Proof. The pair $\langle u, H \rangle$, with $u = u^{(1)} - u^{(2)}$ and $H = H^{(1)} - H^{(2)}$, satisfies

$$\begin{cases} u_t(x, t) + Au(x, t) = F_0(x)H(t), & (x, t) \in \Omega_T, \\ u(x, t) = 0, & (x, t) \in \partial\Omega \times (0, T), \\ u(x, 0) = 0, & x \in \Omega. \end{cases} \quad (21)$$

Using the semigroup theory, we can write the solution $u(x, t)$ of (21) in the following closed form:

$$u(x, t) = \int_0^t H(s)e^{-A(t-s)}F_0(x)ds.$$

Extending the function $H(t)$ by zero outside of $[0, T]$, and assuming $\lambda \geq \lambda_0 > 0$ we get:

$$\int_0^\infty u(x, t)e^{-\lambda t}dt = \int_0^\infty e^{-\lambda t}e^{-At}F_0(x)dt \int_0^\infty e^{-\lambda t}H(t)dt. \quad (22)$$

Using [2, Chapter 1.4] we have

$$w_\lambda := (\lambda I + A)^{-1}F_0 = \int_0^\infty e^{-\lambda t}e^{-At}F_0dt, \quad \forall \lambda \geq \lambda_0.$$

Here the function w_λ is the unique solution to $(\lambda I + A)w_\lambda = F_0$, subject to homogeneous Dirichlet conditions. We would like to employ the strong maximum principle of Hopf to this equation. This principle is valid for differential operators in a nondivergence form. Note that if the coefficients $a_{ij} \in C^1(\Omega)$ then the operator A can be easily transformed into the nondivergence form. We need to know that $w_\lambda \in C^2(\Omega) \cap C(\overline{\Omega})$, which follows from [18, Theorem 6.13].

By the assumption $0 \neq F_0 \geq 0$ we may apply the strong maximum principle ([1, Chapter 6.4], or [18, Theorem 3.5]) for a connected domain to obtain $w_\lambda(x) > 0$, for all $x \in \Omega$. Due to the fact that $u_T(t) := u_T^{(1)}(t) - u_T^{(2)}(t) = 0$ and $H(t) = 0$ for all $t > T$, we see that $u(x, t) = 0$ for all $t > T$ and $x \in \Omega$.

Now, considering (22) at $x = x_0$ and taking into account the condition $g^{(1)}(t) = g^{(2)}(t)$, i.e. $u(x_0, t) = 0$ in $[0, T]$, we get

$$0 = \int_0^T e^{-\lambda t}H(t)dt = \int_0^\infty e^{-\lambda t}H(t)dt, \quad \forall \lambda \geq \lambda_0.$$

Since Laplace transform is one-to-one, we deduce that $H(t) \equiv 0$ in $[0, T]$. Involving this into (21) we arrive at $u(x, t) = 0$ for all $(x, t) \in \Omega_T$, which completes the proof of the theorem. \square

Example 2. Let us consider the same problem as in Example 1, but we set $f = e_n$. Further we assume that the measured output data is given in the form of the value $v_0(t) := u(x_0, t)$ of temperature measured at some interior point $x_0 \in (0, 1)$ for which $f(x_0) \neq 0$. To prove the uniqueness of the solution of this ISP, we assume that $u(x; h_k)$, $u(x_0; h_k) = v_0(t)$, $k = 1, 2$, are two solutions of the inverse problem. Then, by (18) we conclude that the function $v(x) := u(x; h_1) - u(x; h_2)$, satisfies the integral equation

$$\int_0^t e^{-\lambda_n(t-s)}[h_1(s) - h_2(s)]ds = 0, \quad t \in (0, T).$$

First we multiply this by $e^{\lambda_n t}$, then differentiate with respect to the time variable to conclude that $h_1(t) - h_2(t) = 0$, $\forall t \in (0, T)$. So, we can see that the final time measurements are not needed in this case. \square

The following theorem generalizes [Example 2](#) to the more dimensional case with a linear differential operator A of second order, defined by (3). We will employ the maximum/minimum principle for parabolic equations [19,20] in our proof. In this way we do not need the final overdetermination and we allow the function F_0 to change its sign. Another difference between both theorems is in the location of the measurement point x_0 . Its position is arbitrary in [Theorem 2](#) and in [Theorem 3](#) x_0 lies in the support of F_0 . Let us note that the comparison principle is written for a differential operator A in a nondivergence form. To rewrite (3) into the needed form we have to assume that $a_{ij} \in C^1$ for all indices i and j .

Theorem 3. Assume that A is a strongly elliptic operator, $c \geq 0$. Let $\langle u^{(k)}, H^{(k)} \rangle$, $k = 1, 2$, be two solutions of problem (20) such that $H^{(k)}, u^{(k)}, \partial_t u^{(k)}, \partial_{x_i} u^{(k)}, \partial_{x_i x_j}^2 u^{(k)} \in C(\Omega_T)$. Assume that $0 \neq F_0 \in C^2(\bar{\Omega})$, $F_0 = 0$ on Γ , $\Omega \subset \mathbb{R}^n$. If $x_0 \in \Omega$ such that $F_0(x_0) \neq 0$ and $g^{(1)}(t) = g^{(2)}(t)$, then $\langle u^{(1)}, H^{(1)} \rangle = \langle u^{(2)}, H^{(2)} \rangle$ for all $(x, t) \in \Omega_T$.

Proof. Consider the following auxiliary problem

$$\begin{cases} v_t(x, t) + Av(x, t) = 0, & (x, t) \in \Omega_T, \\ v(x, t) = 0, & (x, t) \in \partial\Omega \times (0, T), \\ v(x, 0) = AF_0(x), & x \in \Omega, \end{cases} \quad (23)$$

with the exact solution in a closed form $v(x, t) = e^{-At}AF_0(x)$. The strong minimum/maximum principle [19,20] says that extrema are taken on the parabolic boundary, i.e. $|v(x, t)| \leq C$ for all $(x, t) \in \Omega_T$, because of $F_0 \in C^2(\bar{\Omega})$.

The pair $\langle u, H \rangle$, with $u = u^{(1)} - u^{(2)}$ and $H = H^{(1)} - H^{(2)}$, satisfies (21) and we may write

$$u(x, t) = \int_0^t H(s)e^{-A(t-s)}F_0(x)ds.$$

Applying the operator A to both sides of this relation and taking into account the fact that A commutes with the semigroup, we get

$$H(t)F_0(x) - u_t(x, t) = Au(x, t) = \int_0^t H(s)e^{-A(t-s)}AF_0(x)ds.$$

Considering this equation at the point x_0 and using $g^{(1)}(t) = g^{(2)}(t)$ we get

$$H(t)F_0(x_0) = \int_0^t H(s)e^{-A(t-s)}AF_0(x_0)ds \quad \forall t \in [0, T].$$

Due to the boundedness of the function v , we deduce that

$$|H(t)| |F_0(x_0)| \leq C \int_0^t |H(s)| ds \quad \forall t \in [0, T].$$

Involving the assumption $F_0(x_0) \neq 0$ and the Gronwall lemma we conclude that $H(t) = 0$ in $[0, T]$. This implies that $u(x, t) = 0$ for all $(x, t) \in \Omega_T$. \square

Acknowledgments

The research has partially been supported by the Scientific and Technological Research Council of Turkey (TUBITAK), and through the BOF/GOA-project no. 01G006B7, Ghent University, Belgium.

References

- [1] L.C. Evans, Partial Differential Equations, in: Graduate Studies in Mathematics, vol. 19, American Mathematical Society, Providence, RI, 1998.
- [2] D. Henry, Geometric Theory of Semilinear Parabolic Equations, in: Lecture Notes in Mathematics, vol. 840, Springer-Verlag, New York, 1981.
- [3] M.V. Klibanov, Uniqueness of solutions in the large of some multidimensional inverse problems, in: Non-Classical Problems of Mathematical Physics, Published by Computing Center of the Siberian Branch of the USSR Academy of Science, Novosibirsk, 1981, pp. 101–114. (in Russian).
- [4] M.V. Klibanov, Estimates of initial conditions of parabolic equations and inequalities via lateral Cauchy data, Inverse Problems 22 (2006) 495–514.
- [5] M.V. Klibanov, A.V. Tikhonravov, Estimates of initial conditions of parabolic equations and inequalities in infinite domains via lateral Cauchy data, Journal of Differential Equations 237 (2007) 198–224.
- [6] M.M. Lavrent'ev, V.G. Romanov, S.P. Shishatskii, Ill-Posed Problems of Mathematical Physics and Analysis, American mathematical society, Providence (R.I.), 1986.
- [7] A. Pazy, Semigroup of Linear Operators and Applications to Partial Differential Equations, in: Applied Mathematical Sciences, vol. 44, Springer-Verlag, New York, 1983.
- [8] M. Renardy, J. Rogers, Introduction to Partial Differential Equations, Springer-Verlag, New York, 1993.
- [9] E. Zeidler, Nonlinear Functional Analysis and Its Applications, II/B Nonlinear Monotone Operators, Springer, New York, 1990.
- [10] W. Rundell, Determination of unknown non-homogeneous term in a linear partial differential equation from overspecified boundary data, Applicable Analysis 10 (1980) 231–242.
- [11] A. Hasanov, M. Otelbaev, B. Akpayev, Inverse heat conduction problems with boundary and final time measured output data, Inverse Problems in Science and Engineering 19 (7) (2011) 895–1006.
- [12] A.N. Tikhonov, Théorèmes d'unicité pour l'équation de la chaleur, Mathematical Sbornik 42 (1935) 199–216.

- [13] L. Beilina, M.V. Klibanov, *Approximate Global Convergence and Adaptivity for Coefficient Inverse Problems*, Springer, New York, 2012.
- [14] A.L. Bukhgeim, M.V. Klibanov, Uniqueness in the large of a class of multidimensional inverse problems, *Soviet Mathematics-Doklady* 17 (1981) 244–247.
- [15] V. Isakov, *Inverse Source Problems*, in: *Mathematical Surveys and Monographs*, vol. 34, American Mathematical Society, Providence, RI, 1990.
- [16] V. Isakov, Inverse parabolic problems with the final overdetermination, *Communications on Pure and Applied Mathematics* 54 (1991) 185–209.
- [17] M. Haase, *The Functional Calculus for Sectorial Operators*, in: *Operator Theory: Advances and Applications*, vol. 169, Birkhäuser, 2006.
- [18] D. Gilbarg, N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, in: *A Series of Comprehensive Studies in Mathematics*, vol. 224, Springer-Verlag, New York, 1977.
- [19] A. Friedman, *Partial Differential Equations of Parabolic Type*, Prentice-Hall Inc., 1964.
- [20] W. Walter, On the strong maximum principle for parabolic differential equations, *Proceedings of the Edinburgh Mathematical Society* 29 (1986) 93–96.